But the first three terms of the series, u_0, u_1, u_2, &c., are $u_0 = 1$, $u_1 = 1$, and $u_2 = 2$. Hence, putting $x = o$, we have $Au_1 = 1$, and $Pu_0 = 1$, $\therefore A = -1$ and $P = -1$.

$$\therefore z^2 - z = 1; \therefore r_1 = \frac{1+\sqrt{5}}{2} \text{ and } r_2 = \frac{1-\sqrt{5}}{2}.$$

$$\therefore u_x = C_1\left(\frac{1+\sqrt{5}}{2}\right)^x + C_2\left(\frac{1-\sqrt{5}}{2}\right)^x.$$

When $x = o$ $u_0 = C_1 + C_2 = 1$. And when

$$x = 1\ u_1 = C_1\left(\frac{1+\sqrt{5}}{2}\right) + C_2\left(\frac{1-\sqrt{5}}{2}\right) = 1.$$

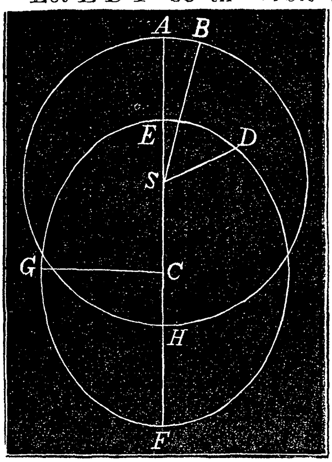
$$\therefore C_1 = \frac{5+\sqrt{5}}{10}, \text{ and } C_2 = \frac{5-\sqrt{5}}{10}.$$

$$\therefore u_x = \frac{(5+\sqrt{5})(1+\sqrt{5})^x + (5-\sqrt{5})(1-\sqrt{5})^x}{10\,(2)^x}.$$

PROBLEM.—TO FIND THE RELATION BETWEEN THE MEAN ANOMALY AND THE TRUE ANOMALY.

BY PROF. JOSEPH FICKLIN, COLUMBIA, MO.

Let EDF be the orbit of the planet, having the sun in the focus at S. Put $a = CE$, $b = CG$, and from S as a center and a radius SA equal to $\sqrt{ab}$, describe the circumference of a circle; then the circle and the ellipse will contain equal areas.



At the same time that the planet departs from E, the perihelion, let a body begin to move with a uniform motion from A through the circumference ABH, and perform a whole revolution in the same time that the planet describes the ellipse. Suppose the body describing the circle to be at B when the planet is at D; the angle ASB is the mean anomaly, and ESD the true anomaly. Put $x =$ the angle ESD, and $\theta = ASB$; then the area of ASB is

$$\frac{\overline{AS}^2}{2}.\theta = \frac{ab}{2}.\theta.$$

Put $r =$ radius-vector of the ellipse, then the area of ESD will be

$$\frac{1}{2}\int r^2 dx; \therefore \frac{ab\theta}{2} = \frac{1}{2}\int r^2 dx, \text{ or } ab\theta = \int r^2 dx \ldots\ldots\ldots (1)$$

The polar equation of the ellipse, the focus being at the pole, is

$$r = \frac{p}{1+e\cos x}; \therefore \int r^2 dx = p^2\int \frac{dx}{(1+e\cos x)^2} \ldots\ldots (2)$$

Assume $\displaystyle\int \frac{dx}{(1+e\cos x)^2} = \frac{A\sin x}{1+e\cos x} + B\int \frac{dx}{1+e\cos x}$.

Taking the differential coefficients, we have

$$\frac{1}{(1+e\cos x)^2} = \frac{A\cos x\,(1+e\cos x)+A\,e\sin^2 x}{(1+e\cos x)^2} + \frac{B(1+e\cos x)}{(1+e\cos x)^2};$$

$\therefore 1 = A\cos x\,(1+e\cos x)+A\,e\sin^2 x+B\,(1+e\cos x)$

$= A\cos x+A\,e\cos^2 x+A\,e\sin^2 x+B+B\,e\cos x=(A+B\,e)\cos x+A\,e+B.$

Equating like powers of $(\cos x)$, we have $A+B\,e=o$, and $A\,e+B=1$; whence

$$B = \frac{1}{1-e^2}, \text{ and } A = -\frac{e}{1-e^2};$$

$$\therefore \int \frac{dx}{(1+e\cos x)^2} = -\frac{e}{1-e^2}\cdot\frac{\sin x}{1+e\cos x} + \frac{1}{1-e^2}\int \frac{dx}{1+e\cos x}.$$

Put $y = \cos x$; then

$$dx = -\frac{dy}{\sqrt{1-y^2}}, \text{ and } \int \frac{dx}{1+e\cos x} = -\int \frac{dy}{(1+e\,y)\sqrt{1-y^2}}$$

Again, put $1-y^2 = (1-y)^2 z^2$;

$$\therefore y = \frac{z^2-1}{z^2+1}, \ dy = \frac{4z\,dz}{(z^2+1)^2}; \therefore -\int \frac{dy}{(1+e\,y)\sqrt{1-y^2}}$$

$$= -2\int \frac{dz}{1-e+z^2(1+e)} = -\frac{2}{1-e}\int \frac{dz}{1+\left(\frac{1+e}{1-e}\right)z^2}$$

$$= -\frac{2}{(1-e)\sqrt{\frac{1+e}{1-e}}}\int \frac{\sqrt{\frac{1+e}{1-e}}\cdot dz}{1+\left(\sqrt{\frac{1+e}{1-e}}\cdot z\right)^2}$$

$$= -\frac{2}{\sqrt{(1+e)(1-e)}}\int\frac{\sqrt{\frac{1+e}{1-e}}\cdot dz}{1+\left(\sqrt{\frac{1+e}{1-e}}\cdot z\right)^2}$$

$$= -\frac{2}{\sqrt{1-e^2}}\int\frac{\sqrt{\frac{1+e}{1-e}}\cdot dz}{1+\left(\sqrt{\frac{1+e}{1-e}}\cdot z\right)^2} = -\frac{2}{\sqrt{1-e^9}}\cdot \operatorname{tang}^{-1}\left(\sqrt{\frac{1+e}{1-e}}\cdot z\right)$$

$\therefore$ area ESD

$$= \frac{1}{2}\int r^2\,dx = -\frac{p^2\,e\sin x}{2(1-e^2)(1+e\cos x)} - \frac{2p^2\cdot\operatorname{tang}^{-1}\left(\sqrt{\frac{1+e}{1-e}}\cdot z\right)}{2(1-e^2)\sqrt{1-e^2}} + C$$

$$= -\frac{p^2\,e\sin x}{2(1-e^2)(1+e\cos x)} - \frac{p^2}{(1-e^2)^{\frac{3}{2}}}\cdot\operatorname{tang}^{-1}\left(\sqrt{\frac{1+e}{1-e}}\cdot z\right) + C$$

$$= -\frac{p^2\,e\sin x}{2(1-e^2)(1+e\cos x)} - \frac{p^2}{(1-e^2)^{\frac{3}{2}}}\cdot\operatorname{tang}^{-1}\left(\sqrt{\frac{1+e}{1-e}\cdot\frac{1+y}{1-y}}\right) + C$$

$$= -\frac{p^2\,e\sin x}{2(1-e^2)(1+e\cos x)} - \frac{p^2}{(1-e^2)^{\frac{3}{2}}}\cdot\operatorname{tang}^{-1}\sqrt{\frac{1+e}{1-e}\cdot\frac{1+\cos x}{1-\cos x}} + C$$

$$= -\frac{p^2\,e\sin x}{2(1-e^2)(1+e\cos x)} - \frac{p^2}{(1-e^2)^{\frac{3}{2}}}\cdot\operatorname{tang}^{-1}\sqrt{\frac{1+e}{1-e}\;\frac{2\cos^2\frac{1}{2}x}{2\sin^2\frac{1}{2}x}} + C$$

$$= -\frac{p^2\,e\sin x}{2(1-e^2)(1+e\cos x)} - \frac{p^2}{(1-e^2)^{\frac{3}{2}}}\cdot\operatorname{tang}^{-1}\left(\sqrt{\frac{1+e}{1-e}}\cdot\cot\tfrac{1}{2}x\right) + C$$

When $x=o$, area $ESD=o$; $\therefore$ from this equation

$$o = -\frac{p^2}{(1-e^2)^{\frac{3}{2}}}\cdot\frac{\pi}{2} + C;\ \text{whence } C = \frac{p^2\,\pi}{2(1-e^2)^{\frac{3}{2}}},$$

and, therefore, the general expression for the area ESD is

$$\frac{1}{2}\int r^2\,dx = -\frac{p^2\,e\sin x}{2(1-e^2)(1+e\cos x)} - \frac{p^2}{(1-e^2)^{\frac{3}{2}}}\cdot\operatorname{tang}^{-1}\left(\sqrt{\frac{1+e}{1-e}}\cot\tfrac{1}{2}x\right) + \frac{p^2\,\pi}{2(1-e^2)^{\frac{3}{2}}}$$

Substituting this value of $\int r^2\,dx$ in (1) we obtain

$$a\,b\,\theta = \frac{p^2\,\pi}{(1-e^2)^{\frac{3}{2}}} - \frac{p^2\,e\sin x}{(1-e^2)(1+e\cos x)} - \frac{2\,p^2}{(1-e^2)^{\frac{3}{2}}}\cdot\operatorname{tang}^{-1}\left(\sqrt{\frac{1+e}{1-e}}\cdot\cot\tfrac{1}{2}x\right);$$

but $p^2 = \frac{b^4}{a^2}$, and $1-e^2 = \frac{b^2}{a^2}$, or, $(1-e^2)^{\frac{3}{2}} = \frac{b^3}{a^3}$; $\therefore \frac{p^2}{(1-e^2)^{\frac{3}{2}}} = \frac{b^4}{a^2} \div \frac{b^3}{a^3}$

$= a\,b$, and the above equation becomes

$$a\,b\,\theta = a\,b\,\pi - \frac{b^2\,e\,\sin x}{1+e\cos x} - a\,b\,\text{tang}^{-1}\left(\sqrt{\frac{1+e}{1-e}}\,.\cot\tfrac{1}{2}x\right),$$

$$\text{or, } a\,\theta = a\,\pi - \frac{b\,e\sin x}{1+e\cos x} - a\,\text{tang}^{-1}\left(\sqrt{\frac{1+e}{1-e}}\,.\cot\tfrac{1}{2}x\right),$$

$$\text{or, } \theta = \pi - \frac{b\,e\sin x}{a(1+e\cos x)} - \text{tang}^{-1}\left(\sqrt{\frac{1+e}{1-e}}\,.\cot\tfrac{1}{2}x\right. \ldots\ldots \quad (3);$$

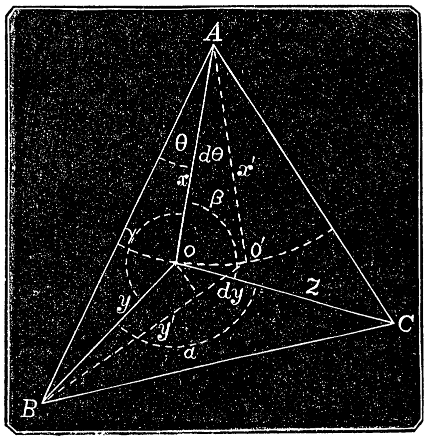
and this is the relation existing between the mean anomaly and the true anomaly.

SOLUTION OF A QUESTION INVOLVING A MINIMUM.

The following problem is found in the *Mathematical Monthly*, (Runkle's):

"Find a point, O, within a triangle, such that $OA^n + OB^n + OC^n =$ a minimum."

[The solution there given is a good one; but a friend of mine, in the United States Coast Survey, dreamed out the solution below, and conveyed it to paper in the morning. I give it for publication; but he disallows the use of his name.]



[*Notation seen from the figure.*]

$x^n + y^n + z^n =$ a minimum.

Let x at first be constant, y and z vary.

$\therefore\ y^{n-1}dy + z^{n-1}dz = o. \qquad (1)$

Now $dy = x\,d\,\theta\sin\gamma$, from infinitesimal triangle

similarly $dz = -x\,d\,\theta\sin\beta$.

Substituting in (1)

$x\,d\,\theta\,(y^{n-1}\sin\gamma - z^{n-1}\sin\beta) = o.$

$\therefore \frac{y^{n-1}}{\sin\beta} = \frac{z^{n-1}}{\sin\gamma}.$

If x also vary we have $\frac{x^{n-1}}{\sin\alpha} = \frac{y^{n-1}}{\sin\beta} = \frac{z^{n-1}}{\sin\gamma}$

If $n=1$, $\alpha=\beta=\gamma$, each $= 120^0$.

If $n=2$, $\frac{x}{\sin\alpha} = \frac{y}{\sin\beta} = \frac{z}{\sin\gamma}$,

that is, the point is the center of gravity of the triangle.

THEO. L. DE LAND, TREAS. DEP'T, WASHINGTON, D. C.